Pergamon

Int. J. Solids Structures Vol. 32, No. 3/4, pp. 383–389, 1995

0020-7683/95 $9.50 + .00

0020-7683(94)00155-3

ON CONTACT PROBLEMS FOR A MEDIUM WITH RIGID FLAT INCLUSIONS OF ARBITRARY SHAPE

G. M. L. GLADWELL
Solid Mechanics Division, Faculty of Engineering, University of Waterloo, Canada N2L 3G1

(*Received* 12 *July* 1994)

Abstract—Consider the elasto-static problem in which the contact between two compressed dissimilar elastic half-spaces is perturbed by disc inclusions of various constant thicknesses, but arbitrary shapes. We show that the interfacial normal displacement is zero where the half-spaces are in contact, and present some generalizations of the result.

1. INTRODUCTION

In a recent paper Selvadurai (1993) considered the problem of a rigid disc inclusion of constant thickness (i.e. a penny) compressed between dissimilar isotropic elastic half-spaces, as shown in Fig. 1. He formulated this axisymmetric problem in terms of Hankel transforms and found, during the course of extensive manipulation, that

(1) the displacements Δ_1, Δ_2 are related by

$$\vartheta_1 \Delta_1 = \vartheta_2 \Delta_2 \tag{1}$$

where $\vartheta_i = \mu_i/(1-\nu_i)$;
(2) the contact stresses on either side of the inclusion are identical;
(3) the normal displacements in the region where the half-spaces are in contact are not only equal, but are separately zero.

Selvadurai's configuration is axisymmetric and it is not immediately clear whether these results, in particular (3), are a consequence of this axisymmetry. Selvadurai's problem involves *complete* contact; the two half-spaces are in contact with the penny over the whole of the upper and lower faces. Gladwell and Hara (1981) had considered the *incomplete* contact problem for a rigid inclusion with faces $z = \pm f_j(r)$, where

$$f_j(r) = D_j - r^2/(2R_j), \quad j = 1,2. \tag{2}$$

It is interesting to note that their Fig. 1 (a sketch) shows the normal displacements equal, but not necessarily zero, where the half-spaces are in contact (for $r \geqslant l$). Later, in commenting on their numerical results they state "Note that the approximate w is not zero, *as it should be*, outside $r = l$". The italics are ours, and indeed on closer inspection we note that w need not be zero outside $r = l$. In fact there is only one easily identifiable case in which w will be zero outside $r = l$; when $\vartheta_1 = \vartheta_2, f_1(r) \equiv f_2(r)$, when the whole configuration is symmetrical about $z = 0$. However, as we will now show, the three results listed above hold for an arbitrarily shaped rigid inclusion of *constant* thickness compressed between two half-spaces.

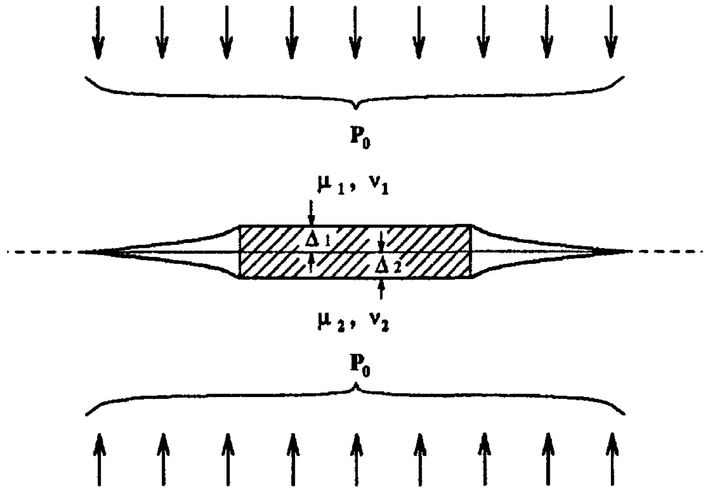

Fig. 1. A rigid disc inclusion of constant thickness compressed between two dissimilar isotropic elastic half-spaces.

2. PAPKOVICH–NEUBER SOLUTIONS

We start with the familiar Papkovich–Neuber (P–N) representation (Gladwell, 1980)

$$2\mu\mathbf{d} = 4(1-\nu)\mathbf{\Psi} - \nabla\{(\mathbf{r}\cdot\mathbf{\Psi})+\Phi\} \tag{3}$$

for the elastic displacement $\mathbf{d} \equiv (u, v, w)$ in a medium with parameters μ, ν. We suppose that there are no body forces, so that $\mathbf{\Psi}$ and Φ are harmonic, i.e.

$$\Delta\mathbf{\Psi} = \mathbf{0}, \quad \Delta\Phi = 0. \tag{4}$$

It is well known that for many regions, of which a half-space is one, it is possible to choose $\mathbf{\Psi}$ so that one of its components, say Ψ_1, is zero. This is achieved by *subtracting* from the given solution ($\mathbf{\Psi}$, Φ) a *null* solution ($\mathbf{\Psi}^0$, Φ^0) having $\Psi_1^0 = \Psi_1$. A null P–N solution is one for which

$$4(1-\nu)\mathbf{\Psi}^0 - \nabla\{(\mathbf{r}\cdot\mathbf{\Psi}^0)+\Phi^0\} \equiv \mathbf{0}. \tag{5}$$

It is easily shown that ($\mathbf{\Psi}^0$, Φ^0) is null if

$$\mathbf{\Psi}^0 = \nabla H \quad \text{where } \Delta H = 0 \tag{6}$$

$$\Phi = 4(1-\nu)H - \nabla(\mathbf{r}\cdot\nabla H). \tag{7}$$

Thus if Ψ_1 is to be annihilated, we must choose H, harmonic, such that

$$\frac{\partial H}{\partial z} = \Psi_1. \tag{8}$$

Barber (1992) shows how this may be done.

In our problem we are concerned with special P–N solutions, for the half-space $z \geqslant 0$ with the surface $z = 0$, i.e. the x, y-plane D, free of shearing stress. In such cases we customarily take

$$\Psi_1 = 0 = \Psi_2, \tag{9}$$

$$(1-2\nu)\Psi_3 = \frac{\partial \Phi}{\partial z}. \tag{10}$$

However, because this choice is crucial to our later argument, we first establish that it is always available, i.e. that if $(\mathbf{\Psi}, \Phi)$ corresponds to an elastic field in $z \geqslant 0$, and

$$\tau_{xz} = 0 = \tau_{yz} \quad (x, y) \in D, \tag{11}$$

then we can find a null field $(\mathbf{\Psi}^0, \Phi^0)$ such that

$$\Psi_1^0 \equiv \frac{\partial H}{\partial x} = \Psi_1, \quad \Psi_2^0 \equiv \frac{\partial H}{\partial y} = \Psi_2. \tag{12}$$

To prove this, we proceed as follows.

Equation (3) shows that the rotation component in the z-direction is

$$\omega_3 = \omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{\vartheta}\left(\frac{\partial \Psi_2}{\partial x} - \frac{\partial \Psi_1}{\partial y}\right). \tag{13}$$

It is harmonic because Ψ_1 and Ψ_2 are harmonic. The stress–strain equations give

$$\frac{\partial \tau_{yz}}{\partial x} - \frac{\partial \tau_{xz}}{\partial y} = \mu\left(\frac{\partial^2 v}{\partial x \partial z} - \frac{\partial^2 u}{\partial y \partial z}\right) = 2\mu \frac{\partial \omega}{\partial z}. \tag{14}$$

Consider the integral

$$J \equiv \int_V (\nabla \omega)^2 \, \mathrm{d}V \tag{15}$$

over the half-space. Since ω is harmonic,

$$\operatorname{div}(\omega \nabla \omega) = \omega \Delta \omega + (\nabla \omega)^2 = (\nabla \omega)^2 \tag{16}$$

and the divergence theorem gives

$$J = -\int_D \omega \frac{\partial \omega}{\partial z} \, \mathrm{d}S; \tag{17}$$

the necessary regularity conditions on ω at infinity are very weak. But eqns (11) and (14) show that

$$\partial \omega / \partial z = 0, \quad (x, y) \in D. \tag{18}$$

Hence $J = 0$, $\nabla \omega \equiv \mathbf{0}$ and $\omega = \text{constant} = 0$ if we take the displacement $\mathbf{d}$ zero at infinity. Equation (13) therefore shows that any P–N solution for the half-space with D free of shear stress must satisfy

$$\frac{\partial \Psi_2}{\partial x} = \frac{\partial \Psi_1}{\partial y}. \tag{19}$$

This is clearly a *necessary* condition for the existence of an H satisfying eqn (12). Now we must find a harmonic H to satisfy eqn (12). To do this, we first construct

$$H_1 = \int_{x_s}^{x} \Psi_1(s, y, z)\,\mathrm{d}s + \int_{y_s}^{y} \Psi_2(x_s, t, z)\,\mathrm{d}t. \tag{20}$$

This satisfies eqn (12), but is not harmonic. However

$$\Delta H_1 = \frac{\partial}{\partial x}\Psi_1(x_s, y_s, z) + \frac{\partial}{\partial y}\Psi_2(x_s, y_s, z) = f(z), \tag{21}$$

so that we need only find $H_2 = H_2(z)$ such that

$$\Delta H_2 = H_2''(z) = -f(z). \tag{22}$$

For this we may take

$$H_2 = \int_{z_s}^{z} (t - z) f(t)\,\mathrm{d}t \tag{23}$$

and then $H = H_1 + H_2$ is the required harmonic function.

We note that all the operations described, such as the integrations in eqns (20) and (23) are unambiguous in the half-space. We have established eqn (9); eqn (10) now follows easily from the explicit expressions for τ_{xz}, τ_{yz} in terms of Ψ_3 and Φ.

3. ANALYSIS OF THE CONTACT PROBLEM

Suppose that instead of being circular, the inclusion occupies a region D_1 in the x, y-plane, there is a separation zone D_2, and the half-spaces are in contact over the remainder, D_3, of the x, y-plane D.

The elastic solution for each half-space may be obtained as the superposition of two fields. The first corresponds to a uniform stress field in the z-direction, namely $\tau_{zz}^{(j)} = -p_0$, for which the normal displacments are given by

$$2(1+\nu_j)\mu_j w_j = -p_0 z, \quad j = 1,2. \tag{24}$$

The second may be obtained from a P–N solution

$$2\mu_j \mathbf{d}_j = 4(1-\nu_j)\mathbf{\Psi}^{(j)} - \nabla\{(\mathbf{r}\cdot\mathbf{\Psi}^{(j)}) + \Phi^{(j)}\}, \quad j = 1,2 \tag{25}$$

where $\mathbf{d}_j \equiv (u_j, v_j, w_j)$ is the vector of elastic displacements in the half-spaces (1) $z \geqslant 0$, (2) $z \leqslant 0$. Since there are no body forces present, the potentials $\mathbf{\Psi}^{(j)}$ and $\Phi^{(j)}$ are harmonic.

There are no shearing stresses on the interface, so that

$$\tau_{xz}^{(j)} = 0 = \tau_{yz}^{(j)}, \quad x, y \in D, \quad j = 1,2. \tag{26}$$

The analysis of section 2 shows that we may take $\mathbf{\Psi}^{(j)} = (0, 0, \Psi_j)$ and

$$(1-2\nu_j)\Psi_j = \partial\Phi^{(j)}/\partial z, \quad j = 1, 2. \tag{27}$$

This yields

$$2\mu_j w_j = 2(1-\nu_j)\Psi_j - z\frac{\partial \Psi_j}{\partial z}, \quad \tau_{zz}^{(j)} = \frac{\partial \Psi_j}{\partial z} - z\frac{\partial^2 \Psi_j}{\partial z^2}, \quad j = 1,2, \tag{28}$$

so that the boundary conditions for the Ψ_j are

$$\Psi_1 = \vartheta_1 \Delta_1, \quad \Psi_2 = -\vartheta_2 \Delta_2 \quad (x,y) \in D_1 \tag{29}$$

$$\frac{\partial \Psi_j}{\partial z} = p_0 \quad (x,y) \in D_2 \tag{30}$$

$$\frac{\Psi_1}{\vartheta_1} - \frac{\Psi_2}{\vartheta_2} = 0 \quad (x,y) \in D_3 \tag{31}$$

$$\frac{\partial \Psi_1}{\partial z} = \frac{\partial \Psi_2}{\partial z} \quad (x,y) \in D_3. \tag{32}$$

In addition there is the equilibrium condition for the inclusion, namely

$$\int_{D_1} \left(\frac{\partial \Psi_1}{\partial z} - \frac{\partial \Psi_2}{\partial z}\right) \mathrm{d}S = 0 \tag{33}$$

and the regularity conditions at infinity, namely

$$\Psi_j(x,y,z) = O((x^2+y^2+z^2)^{-1/2}). \tag{34}$$

The potential Ψ_1 is harmonic in the upper half-space, and Ψ_2 in the lower half-space. Define

$$\Psi_2^*(x,y,z) = -\Psi_2\,(x,y,-z) \tag{35}$$

so that Ψ_2^* is harmonic in the upper half-space V. The conditions (29)–(33) translate into conditions for Ψ_1, Ψ_2^* on D, namely

$$\Psi_1 = \vartheta_1 \Delta_1, \quad \Psi_2^* = \vartheta_2 \Delta_2 \quad (x,y) \in D_1 \tag{36}$$

$$\frac{\partial \Psi_1}{\partial z} = p_0 = \frac{\partial \Psi_2^*}{\partial z} \quad (x,y) \in D_2 \tag{37}$$

$$\frac{\Psi_1}{\vartheta_1} + \frac{\Psi_2^*}{\vartheta_2} = 0 \quad (x,y) \in D_3 \tag{38}$$

$$\frac{\partial \Psi_1}{\partial z} = \frac{\partial \Psi_2^*}{\partial z} \quad (x,y) \in D_3 \tag{39}$$

$$\int_{D_1} \left(\frac{\partial \Psi_1}{\partial z} - \frac{\partial \Psi_2^*}{\partial z}\right) \mathrm{d}S = 0. \tag{40}$$

Consider the integral

$$I \equiv \int_V \operatorname{div}\{(\Psi_1 - \Psi_2^*)\,\nabla(\Psi_1 - \Psi_2^*)\}\,\mathrm{d}V \tag{41}$$

taken over the upper half-space. Since $\Psi_1 - \Psi_2^*$ is harmonic in V, we have

$$I = \int_V \{\nabla(\Psi_1 - \Psi_2^*)\}^2 \, dV. \tag{42}$$

On the other hand, Gauss' theorem gives

$$I = -\int_D (\Psi_1 - \Psi_2^*) \frac{\partial}{\partial z} (\Psi_1 - \Psi_2^*) \, dS. \tag{43}$$

The conditions (37)–(39) show that there is no contribution to I from D_2 or D_3. On D_1 both Ψ_1 and Ψ_2^* are constant so that

$$I = -(\vartheta_1 \Delta_1 - \vartheta_2 \Delta_2) \int_{D_1} \frac{\partial}{\partial z} (\Psi_1 - \Psi_2^*) \, dS \tag{44}$$

and this is zero because of the equilibrium condition (41). Thus $I \equiv 0$, and hence, from eqn (42), $\Psi_1 - \Psi_2^* = \text{const}$ and the regularity condition (34) shows that this must be zero. Therefore

$$\Psi_1(x, y, z) = \Psi_2^*(x, y, z) = -\Psi_2(x, y, -z) \tag{45}$$

from which the results (1)–(3) immediately follow.

4. GENERALIZATIONS

In this argument we have implicitly considered the region D_1 occupied by the inclusion to be connected, so that there is just one equilibrium condition (40). However, D_1 may be made up of a number of connected regions, so that

$$D_1 = \bigcup_{i=1}^{n} D_{1i} \tag{46}$$

and the inclusions may be of different thicknesses. Then the boundary conditions are

$$\Psi_1 = \vartheta_1 \Delta_{1i}, \quad \Psi_2^* = \vartheta_2 \Delta_{2i} \quad (x, y) \in D_{1i} \tag{47}$$

and the equilibrium conditions are

$$\int_{D_{1i}} \left(\frac{\partial \Psi_1}{\partial z} - \frac{\partial \Psi_2^*}{\partial z} \right) dS = 0, \quad i = 1,2,\ldots,n. \tag{48}$$

Again the argument holds and result (1) generalizes to

$$\vartheta_1 \Delta_{1i} = \vartheta_2 \Delta_{2i}, \quad i = 1,2,\ldots,n. \tag{49}$$

while (2) holds for each inclusion and (3) still holds.

Even though the results hold in this generalized configuration, a reviewer correctly commented that it is difficult to visualize a situation in which two inclusions of different thickness near to each other could maintain contact over all D_1. For an example, we can consider two circular disc inclusions of different thicknesses which are almost in contact with each other; separation will occur over some part of the thinner inclusion.

To generalize the result further we note that the integral I in eqn (43) may be written as in eqn (44) provided that $\Psi_1 - \Psi_2^*$ is constant on D_1; Ψ_1 and Ψ_2^* do not have to be separately zero. Again I will be zero and eqn (45) will follow. To use this we note that if the faces of the inclusion are plane then we may apply a rigid body rotation, of which the z-component is

$$w = \alpha x + \beta y, \tag{50}$$

to the whole space, to make $\Psi_1 - \Psi_2^*$ constant on D_1. (We can always do this provided that D_1 is connected; if it is made up of connected regions D_{1i}, we can do it provided that the plane faces of the rigid inclusions in contact with the half-space $z > 0$ are all parallel, as are those in contact with the half-space $z < 0$.) Thus the inclusion(s) will align themselves so that, in the new frame of reference

$$\vartheta_1 w_{1i} = -\vartheta_2 w_{2i} \quad (x, y) \in D_{1i} \tag{51}$$

while results (2) and (3) listed in section 1 will still hold, the latter again in the new frame of reference. If the faces of the plane depart from planes, then the contact between them and the half-spaces will be incomplete and the analysis will fail.

Acknowledgement—I am indebted to James R. Barber for suggesting the last generalization, and for making many insightful comments on an earlier version of this paper. He also pointed out that the whole analysis may be generalized to the case of two transversely isotropic half-spaces.

REFERENCES

Barber, J. R. (1992). *Elasticity*. Kluwer, Dordrecht.

Gladwell, G. M. L. (1980). *Contact Problems in the Classical Theory of Elasticity*. Sijthoff and Noordhoff, The Netherlands.

Gladwell, G. M. L. and Hara, T. (1981). The contact problem for a rigid obstacle pressed between two dissimilar elastic half-spaces. *Q. J. Mech. Appl. Math.* **34**, 251–263.

Selvadurai, A. P. S. (1993). A unilateral contact problem for a rigid disc inclusion embedded between two dissimilar halfspaces. *Q. J. Mech. Appl. Math.* (in press).